

# The elementary 3-Kronecker modules

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**Abstract.** The 3-Kronecker quiver has two vertices, namely a sink and a source, and 3 arrows. A regular representation of a representation-infinite quiver such as the 3-Kronecker quiver is said to be elementary provided it is non-zero and not a proper extension of two regular representations. Of course, any regular representation has a filtration whose factors are elementary, thus the elementary representations may be considered as the building blocks for obtaining all the regular representations. We are going to determine the elementary 3-Kronecker modules. It turns out that all the elementary modules are combinatorially defined.

Let  $k$  be an algebraically closed field and  $Q = K(3)$  the 3-Kronecker quiver

$$1 \rightrightarrows 2$$

The *dimension vector* of a representation  $M$  of  $Q$  is the pair  $(\dim M_1, \dim M_2)$ .

We denote by  $A$  the arrow space of  $Q$ , it is a three-dimensional vector space, thus  $\Lambda = \begin{bmatrix} k & A \\ 0 & k \end{bmatrix}$  is the path algebra of  $Q$ . Note that  $\Lambda$  is a finite-dimensional  $k$ -algebra which is connected, hereditary and representation-infinite. The  $\Lambda$ -modules will be called *3-Kronecker modules*. Of course, choosing a basis of  $A$ , the 3-Kronecker modules are just the representations of  $K(3)$ .

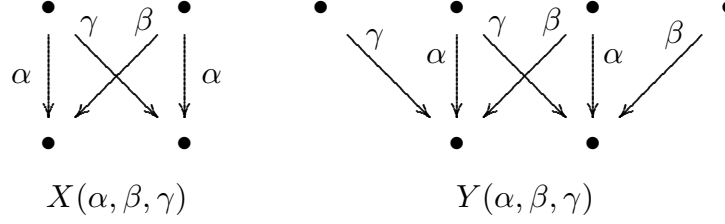
**Elementary modules.** In general, if  $\Lambda$  is a finite-dimensional  $k$ -algebra, we denote by  $\text{mod } \Lambda$  the category of all (finite-dimensional left)  $\Lambda$ -modules. We denote by  $\tau$  the Auslander-Reiten translation in  $\text{mod } \Lambda$ .

Now let  $\Lambda$  be the path algebra of a finite acyclic quiver. A  $\Lambda$ -module  $M$  is said to be *preprojective* provided there are only finitely many isomorphism classes of indecomposable modules  $X$  with  $\text{Hom}(X, M) \neq 0$ , or, equivalently, provided  $\tau^t M = 0$  for some natural number  $t$ . Dually,  $M$  is said to be *preinjective* provided there are only finitely many isomorphism classes of indecomposable modules  $X$  with  $\text{Hom}(M, X) \neq 0$ , or, equivalently, provided  $\tau^{-t} M = 0$  for some natural number  $t$ . A  $\Lambda$ -module  $M$  is said to be *regular* provided it has no indecomposable direct summand which is preprojective or preinjective.

A regular  $\Lambda$ -module  $M$  is said to be *elementary* provided there is no short exact sequence  $0 \rightarrow M' \rightarrow M'' \rightarrow 0$  with  $M', M''$  being non-zero regular modules (the definition is due to Crawley-Boevey, for basic results see Kerner and Lukas [L,KL,K]) and the appendix 1. Of course, any regular module has a filtration whose factors are elementary. If  $M$  is elementary, then all the modules  $\tau^t M$  with  $t \in \mathbb{Z}$  are elementary.

The aim of this note is to determine the elementary 3-Kronecker modules. Let  $\alpha, \beta, \gamma$  be a basis of  $A$ . Let  $X(\alpha, \beta, \gamma)$  and  $Y(\alpha, \beta, \gamma)$  be the  $\Lambda$ -module defined by the following

pictures:



Here, we draw a corresponding coefficient quiver and require that all non-zero coefficients are equal to 1. Thus, for example  $X(\alpha, \beta, \gamma) = (k^2, k^2; \alpha, \beta, \gamma)$  with  $\alpha(a, b) = (a, b)$ ,  $\beta(a, b) = (b, 0)$  and  $\gamma(a, b) = (0, a)$  for  $a, b \in k$ .

**Theorem.** *The dimension vectors of the elementary 3-Kronecker modules are the elements in the  $\tau$ -orbits of  $(1, 1)$ ,  $(2, 1)$ ,  $(2, 2)$  and  $(4, 2)$ .*

*Any indecomposable representation with dimension vector in the  $\tau$ -orbit of  $(1, 1)$  and  $(2, 1)$  is elementary.*

*An indecomposable representation with dimension vector  $(2, 2)$  or  $(4, 2)$  is elementary if and only if it is of the form  $X(\alpha, \beta, \gamma)$  or  $Y(\alpha, \beta, \gamma)$ , respectively for some basis  $\alpha, \beta, \gamma$  of  $A$ .*

The indecomposable representations with dimension vectors in the  $\tau$ -orbits of  $(1, 1)$  and  $(2, 1)$  have been studied in several papers. They are the even index Fibonacci modules, see [FR2, FR3, R4]. If  $M$  is indecomposable and  $\dim M = (1, 1)$  or  $(2, 1)$ , then there is a basis  $\alpha, \beta, \gamma$  of  $A$  such that  $M = B(\alpha)$  or  $M = V(\beta, \gamma)$ , respectively, defined as follows:



Note that  $B(\alpha)$  is the unique indecomposable 3-Kronecker module of length 2 which is annihilated by  $\beta$  and  $\gamma$ , whereas  $V(\beta, \gamma)$  is the unique indecomposable 3-Kronecker module of length 3 with simple socle which is annihilated by  $\alpha$ .

The indecomposable modules with dimension vector  $(1, 1)$  are called *bristles* in [R3]. The indecomposable representations with dimension vector  $(2, 1)$  have been considered in [BR]: there, it has been shown that any arrow  $\alpha$  of a quiver gives rise to an Auslander-Reiten sequence with indecomposable middle term say  $M(\alpha)$ ; in this way, we obtain the sequence:

$$0 \rightarrow V(\beta, \gamma) \rightarrow M(\alpha) \rightarrow \tau^- V(\beta, \gamma) \rightarrow 0.$$

The study of the  $\tau$ -orbits of the indecomposable 3-Kronecker modules with dimension vectors  $(1, 1)$  and  $(2, 1)$  in the papers [FR2, FR3, R4, R5] uses the universal covering  $\tilde{K}(3)$  of the Kronecker quiver  $K(3)$ . The quiver  $\tilde{K}(3)$  is the 3-regular tree with bipartite orientation. Since the 3-Kronecker modules  $B(\alpha)$  and  $V(\beta, \gamma)$  are cover-exceptional (they are push-downs of exceptional representations of  $\tilde{K}(3)$ ), it follows that all the modules in the  $\tau$ -orbits of  $B(\alpha)$  and  $V(\beta, \gamma)$  are cover-exceptional, and therefore tree modules in the sense of [R2].

In general, one should modify the definition of a tree module as follows: Let  $Q$  be any quiver. For any pair of vertices  $x, y$  of  $Q$ , let  $A(x, y)$  be the corresponding arrow space, this is the vector space with basis the arrows  $x \rightarrow y$ . If  $\alpha(1), \dots, \alpha(t)$  are the arrows  $x \rightarrow y$  and  $\beta = \sum a_i \alpha(i)$  with all  $a_i \in k$  is an element of  $A(x, y)$ , we may consider for any representation  $M = (M_x, M_\alpha)_{x \in Q_0, \alpha \in Q_1}$  the linear combination  $M_\beta = \sum a_i M_{\alpha(i)}$ . Given a basis  $\mathcal{B}(x, y)$  of the arrow space  $A(x, y)$ , for all vertices  $x, y$  of  $Q$  as well as a basis  $\mathcal{B}(M, x)$  of the vector space  $M_x$ , for all vertices  $x$  of  $Q$ , we may write the linear maps  $M_b$  with  $b \in \mathcal{B}(x, y)$  as matrices with respect to the bases  $\mathcal{B}(M, x), \mathcal{B}(M, y)$ . Looking at these matrices, we obtain a coefficient quiver  $\Gamma(\mathcal{B}(x, y), \mathcal{B}(M, x))$  as in [R2]. A representation  $M$  of the path algebra  $kQ$  should be called a *tree module* provided  $M$  is indecomposable and there are bases  $\mathcal{B}(x, y)$  of the arrow spaces  $A(x, y)$  and  $\mathcal{B}(M, x)$  of the vector spaces  $M_x$  such that  $\Gamma(\mathcal{B}(x, y), \mathcal{B}(M, x))$  is a tree. Of course, in case  $Q$  has no multiple arrows, this coincides with the definition given in [R2]. But in general, we now allow base changes in the arrow spaces. Note that such base changes in the arrow spaces do not effect the  $kQ$ -module  $M$ , but only its realization as the representation of a quiver. There is the following interesting consequence: Any indecomposable representation of the 2-Kronecker quiver is a tree module, see Appendix 2. Using this modified definition, we see immediately that *all the indecomposable modules with dimension vector in the  $\tau$ -orbits of  $(1, 1)$  and  $(2, 1)$  are tree modules*. On the other hand, the modules  $X(\alpha, \beta, \gamma)$  (and also  $Y(\alpha, \beta, \gamma)$ ) are not tree modules, see Lemma 4.2.

Whereas the modules in the  $\tau$ -orbits of the elementary 3-Kronecker modules with dimension vectors  $(1, 1)$  and  $(2, 1)$  are quite well understood, a similar study of those in the  $\tau$ -orbits of modules with dimension vectors  $(2, 2)$  and  $(4, 2)$  is missing. It seems that any module  $M$  in these  $\tau$ -orbits has a coefficient quiver with a unique cycle. A first structure theorem for these modules is exhibited in section 5.

We say that an element  $(x, y) \in K_0(\Lambda) = \mathbb{Z}^2$  is *non-negative* provided  $x, y \geq 0$ . The non-negative elements in  $K_0(\Lambda)$  are just the possible dimension vectors of  $\Lambda$ -modules. Note that  $K_0(\Lambda)$  is endowed with the quadratic form  $q$  defined by  $q(x, y) = x^2 + y^2 - 3xy$  (see for example [R1]). A dimension vector  $\mathbf{d}$  is said to be *regular* provided  $q(\mathbf{d}) < 0$ . There are precisely two  $\tau$ -orbits of dimension vectors  $\mathbf{d}$  with  $q(\mathbf{d}) = -1$ , namely the  $\tau$ -orbits of  $(1, 1)$  and  $(2, 1)$ . Similarly, there are precisely two  $\tau$ -orbits of dimension vectors  $\mathbf{d}$  with  $q(\mathbf{d}) = -4$ , namely the  $\tau$ -orbits of  $(2, 2)$  and  $(4, 2)$ . The remaining regular dimension vectors  $\mathbf{d}$  satisfy  $q(\mathbf{d}) \leq -5$ .

**Corollary.** *Let  $\Lambda = kK(3)$  and  $(x, y)$  a dimension vector. There exists an elementary module  $M$  with dimension vector  $(x, y)$  if and only if  $q(x, y)$  is equal to  $-1$  or  $-4$ .*

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## 1. The BGP-shift $\sigma$ .

Let  $\sigma$  denote the BGP-shift of  $K_0(\Lambda) = \mathbb{Z}^2$  given by  $\sigma(x, y) = (3x - y, x)$ , and let  $\tau = \sigma^2$ .

We denote by  $\sigma, \sigma^-$  the *BGP-shift functors* for  $\text{mod } \Lambda$  (they correspond to the reflection functors of Bernstein-Gelfand-Ponomarev in [BGP], but take into account that the opposite of the 3-Kronecker quiver is again the 3-Kronecker quiver). If  $M = (M_1, M_2; \alpha, \beta, \gamma)$  is a representation of  $Q$ , we denote by  $(\sigma M)_1$  the kernel of the map  $[\alpha \ \beta \ \gamma] : M_1^3 \rightarrow M_2$  and put  $(\sigma M)_2 = M_1$ ; the maps  $\alpha, \beta, \gamma : (\sigma M)_1 \rightarrow (\sigma M)_2$  are given by the corresponding projections. Similarly,  $(\sigma^- M)_2$  is the cokernel of the map  $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} : M_1 \rightarrow M_2^3$  and we put  $(\sigma^- M)_1 = M_2$ ; now the maps  $\alpha, \beta, \gamma : (\sigma^- M)_1 \rightarrow (\sigma^- M)_2$  are just the corresponding restrictions. Note that  $\sigma^2$  is just the Auslander-Reiten translation  $\tau$  (we should stress that this relies on the fact that we deal with a quiver without cyclic walks of odd length, see [G]).

**Remark.** The functors  $\sigma$  and  $\sigma^-$  depend on the choice of the basis  $\alpha, \beta, \gamma$  of  $A$ , thus we should write  $\sigma = \sigma_{\alpha, \beta, \gamma}$  and  $\sigma^- = \sigma_{\alpha, \beta, \gamma}^-$ .

If  $N$  is an indecomposable representation of  $K(3)$  different from  $S(2)$ , then  $\mathbf{dim} \sigma N = \sigma \mathbf{dim} N$ ; similarly, if  $N$  is indecomposable and different from  $S(1)$ , then  $\mathbf{dim} \sigma^- N = \sigma^- \mathbf{dim} N$  (here,  $S(1)$  and  $S(2)$  are the simple representations of  $K(3)$ ; they are defined by  $\mathbf{dim} S(1) = (1, 0)$ ,  $\mathbf{dim} S(2) = (0, 1)$ ).

An indecomposable  $\Lambda$ -module  $M$  is regular if and only if all the modules  $\sigma^n N$  and  $\sigma^{-n} N$  with  $n \in \mathbb{N}$  are nonzero. The restriction of  $\sigma$  to the full subcategory of all regular modules is a self-equivalence with inverse  $\sigma^-$  and a regular module  $M$  is elementary if and only if  $\sigma M$  is elementary. We say that an indecomposable representation  $M$  of  $K(3)$  is of  $\sigma$ -type  $(x, y)$  provided  $\mathbf{dim} M$  belongs to the  $\sigma$ -orbit of  $(x, y)$ .

In terms of  $\sigma$ , the main result can be formulated as follows:

**Theorem.** *The elementary  $kK(3)$ -modules are of  $\sigma$ -type  $(1, 1)$  and  $(2, 2)$ . All the indecomposable representations of  $\sigma$ -type  $(1, 1)$  are elementary and tree modules. An indecomposable representation of  $\sigma$ -type  $(2, 2)$  is either elementary or else a tree module.*

The tree modules with dimension vector  $(2, 2)$  are precisely the representations of the form



for some basis  $\alpha, \beta, \gamma$  of  $A$ .

## 2. Reduction to the dimension vectors $(x, y)$ with $\frac{2}{3}x \leq y \leq x$ .

Let us denote by  $\mathbf{R}$  the set of regular dimension vectors. As we have mentioned,  $\sigma$  maps  $\mathbf{R}$  onto  $\mathbf{R}$ . There is the additional transformation  $\delta$  on  $K_0(\Lambda)$  defined by  $\delta(x, y) = (y, x)$ . Of course, it also sends  $\mathbf{R}$  onto  $\mathbf{R}$ . If  $M$  is a representation of  $Q(3)$ , then  $\delta(\mathbf{dim} M) = \mathbf{dim} M^*$ , where  $M^*$  is the dual representation of  $M$  (defined in the obvious way:  $(M^*)_1$  is the  $k$ -dual of  $M_2$ ,  $(M^*)_2$  is the  $k$ -dual of  $M_1$ , the map  $(M^*)_\alpha$  is the  $k$ -dual of  $M_\alpha$ , and so on).

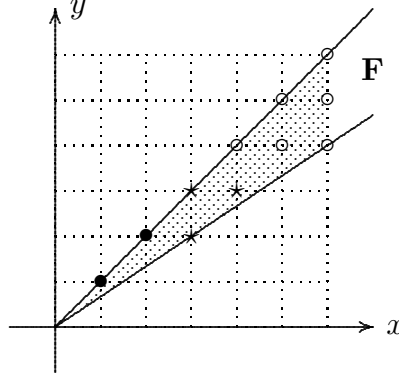
**Lemma.** *The subset*

$$\mathbf{F} = \{(x, y) \mid \frac{2}{3}x \leq y \leq x\}$$

*is a fundamental domain for the action of  $\sigma$  and  $\delta$  on  $\mathbf{R}$ .*

The proof is easy. Let us just mention that  $\sigma(3, 2) = (2, 3)$  and that for  $(x, y) \in \mathbf{R}$  with  $\sigma(x, y) = (x', y')$ , we have  $\frac{y}{x} > \frac{y'}{x'}$  (this condition explains why we call  $\sigma$  a shift).  $\square$ .

It follows that for looking at an elementary module, we may use the shift  $\sigma$  and duality in order to obtain an elementary module  $M$  with  $\mathbf{dim} M \in \mathbf{F}$ . Here is the set  $\mathbf{F}$ :



In the next section 3, we first will consider the pairs  $(x, y) \in \mathbf{F}$  with  $y \geq 4$ , they are marked by a circle  $\circ$ . Then we deal with the three special pairs  $(3, 2)$ ,  $(3, 3)$  and  $(4, 3)$  marked by a star  $\star$  (actually, instead of  $(4, 3)$  and  $(3, 2)$ , we will look at  $(3, 4)$  and  $(2, 3)$ , respectively). As we will see in section 3, all these pairs cannot occur as dimension vectors of elementary modules.

As a consequence, the only possible dimension vectors in  $\mathbf{F}$  which can occur as dimension vectors of elementary modules are  $(1, 1)$  and  $(2, 2)$ ; they are marked by a bullet  $\bullet$  and will be studied in section 4.

### 3. Dimension vectors without elementary modules.

**Lemma 3.1.** *Assume that  $M$  is a regular module with a proper non-zero submodule  $U$  such that both dimension vectors  $\mathbf{dim} U$  and  $\mathbf{dim} M/U$  are regular. Then  $M$  is not elementary.*

*Proof.* This is a direct consequence of the fact that  $M$  is elementary if and only if for any submodule  $U$  the submodule  $U$  is preprojective or the factor module  $M/U$  is preinjective, see the Appendix 1.

**Lemma 3.2.** *A 3-Kronecker module  $M$  with  $\mathbf{dim} M = (x, y)$  such that  $2 \leq y \leq x + 1$  has a submodule  $U$  with dimension vector  $(1, 2)$ .*

*Proof.* Let us show that there are non-zero elements  $m \in M_1$  and  $\alpha \in A$  such that  $\alpha m = 0$ . The multiplication map  $A \otimes_k M_1 \rightarrow M_2$  is a linear map, let  $W$  be its kernel. Since  $\dim A = 3$ , we see that  $\dim A \otimes_k M_1 = 3x$ . Since  $\dim M_2 = y$ , it follows that  $\dim W \geq 3x - y$ . The projective space  $\mathbb{P}(A \otimes M_1)$  has dimension  $3x - 1$ , the decomposable tensors

in  $A \otimes M_1$  form a closed subvariety  $\mathcal{V}$  of  $\mathbb{P}(A \otimes M_1)$  of dimension  $(3-1) + (x-1) = x+1$ . Since  $\mathcal{W} = \mathbb{P}(V)$  is a closed subspace of  $\mathbb{P}(A \otimes M_1)$  of dimension  $3x-y-1$ , it follows that

$$\dim(\mathcal{V} \cap \mathcal{W}) \geq (x+1) + (3x-y-1) - (3x-1) = x-y+1.$$

By assumption,  $x-y+1 \geq 0$ , thus  $\mathcal{V} \cap \mathcal{W}$  is non-empty. As a consequence, we get non-zero elements  $m \in V, \alpha \in A$  such that  $\alpha m = 0$ , as required.

Given non-zero elements  $m \in M_1$  and  $\alpha \in A$  such that  $\alpha m = 0$ , the element  $m$  generates a submodule  $U'$  which is annihilated by  $\alpha$ , thus  $\mathbf{dim} U' = (1, u)$  with  $0 \leq u \leq 2$ . Since  $y \geq 2$ , there is a semi-simple submodule  $U''$  of  $M$  with dimension vector  $(0, 2-u)$  such that  $U' \cap U'' = 0$ . Let  $U = U' \oplus U''$ . This is a submodule of  $M$  with dimension vector  $\mathbf{dim} U = \mathbf{dim} U' \oplus U'' = (1, 2)$ .

Remark. Under the stronger assumption  $2 \leq y < x$ , we can argue as follows: We have  $\langle (1, 2), (x, y) \rangle = x + 2y - 3y = x - y > 0$ , where  $\langle -, - \rangle$  is the canonical bilinear form on  $K_0(\Lambda)$  (see [R1]), thus  $\text{Hom}(N, M) \neq 0$  for any module  $N$  with  $\mathbf{dim} N = (1, 2)$ . The image of any non-zero map  $f: N \rightarrow M$  has dimension vector  $(1, u)$  with  $0 \leq u \leq 2$ .

**Lemma 3.3.** *If  $(x, y) \in \mathbf{F}$  and  $y \geq 4$ , then  $(x-1, y-2)$  is a regular dimension vector.*

Proof. Since  $y \leq x$ , we have  $y-2 \leq x-1$ . On the other hand, the inequalities  $y \geq 4$  and  $y \geq \frac{2}{3}x$  imply the inequality  $y-2 \geq \frac{2}{5}(x-1)$ . Thus  $\frac{2}{5}(x-1) \leq y-2 \leq x-1$ . As a consequence,  $(x-1, y-2)$  is a regular dimension vector.

We are now able to provide a proof for the first assertion of the Theorem: *The elementary  $kK(3)$ -modules are of  $\sigma$ -type  $(1, 1)$  and  $(2, 2)$ .*

Proof. Let  $M$  be elementary with dimension vector  $\mathbf{dim} M = (x, y) \in \mathbf{F}$ . First, assume that  $y \geq 4$ . According to Lemma 3.2, there is a submodule  $U$  with the regular dimension vector  $\mathbf{dim} U = (1, 2)$ . The factor module  $M/U$  has dimension vector  $(x-1, y-2)$  and  $(x-1, y-1)$ . According to Lemma 3.3, also  $(x-1, y-2)$  is a regular dimension vector. Using Lemma 3.1, we obtain a contradiction.

It remains to show that the dimension vectors  $(3, 2), (3, 3), (4, 3)$  cannot occur. Using duality, we may instead deal with the dimension vectors  $(2, 3), (3, 3), (3, 4)$ . Thus, assume there is given an elementary module  $N$  with dimension vector  $(2, 3), (3, 3)$  or  $(3, 4)$ . According to Lemma 3.2, it has a submodule  $U$  with dimension vector  $(1, 2)$ . The corresponding factor module  $M/U$  has dimension vector  $(1, 1), (2, 1), (2, 2)$ , respectively. But all these dimension vectors are regular. Again Lemma 3.1 provides a contradiction.  $\square$ .

#### 4. The indecomposable modules with dimension vector $(1, 1)$ and $(2, 2)$ .

**Dimension vector  $(1, 1)$ .** *Any indecomposable  $\Lambda$ -module  $M$  with dimension vector  $(1, 1)$  is of the form*

$$\begin{array}{c} \bullet \\ \alpha \downarrow \\ \bullet \end{array}$$

for some basis  $\alpha, \beta, \gamma$  of  $A$ , thus a tree module. Namely,  $M = P(1)/U$ , where  $P(1)$  is the indecomposable projective module corresponding to the vertex 1 and  $U$  is a two-dimensional submodule of  $P(1)$ . Actually, we may consider  $U$  as a two-dimensional subspace of  $P(1)$ . Let  $\alpha, \beta, \gamma$  be a basis of  $A$  such that  $U = \langle \beta, \gamma \rangle$ .

Of course, any indecomposable  $\Lambda$ -module with dimension vector  $(1, 1)$  is elementary.

### The indecomposable $\Lambda$ -modules with dimension vector $(2, 2)$ .

**Lemma 4.1.** *An indecomposable module with dimension vector  $(2, 2)$  is elementary if and only if it is of the form  $X(\alpha, \beta, \gamma)$ .*

Proof. First we show: *The modules  $M = X(\alpha, \beta, \gamma)$  are elementary.* We have to verify that any non-zero element of  $M_1$  generates a 3-dimensional submodule. We see this directly for the elements  $(1, 0)$  and  $(0, 1)$  of  $M_1 = k^2$ . If  $(a, b)$  with  $a \neq 0, b \neq 0$ , then  $\beta(a, b) = (b, 0)$  and  $\gamma(a, b) = (0, a)$  are linearly independent elements of  $M_2 = k^2$ . This completes the proof.

Conversely, let  $M$  be an elementary module with dimension vector  $(2, 2)$ . Let us show that the restriction of  $M$  to any 2-Kronecker subalgebra has 2-dimensional endomorphism ring. Let  $\alpha, \beta, \gamma$  be a basis of the arrow space and consider the restriction  $M'$  of  $M$  to the subquiver  $K(2)$  with basis  $\beta, \gamma$ . If  $M'$  has a simple injective direct summand, then either  $M'$  is annihilated by  $\alpha$ , then  $M'$  is a simple injective submodule of  $M$ , therefore  $M$  is not indecomposable, impossible. If  $M'$  is not annihilated by  $\alpha$ , then  $M' + \alpha(M')$  is an indecomposable submodule of dimension 2, thus  $M$  is not elementary. Dually,  $M'$  has no simple projective direct summand. It remains to exclude the case that  $M' = R \oplus R$  for some simple regular representation  $R$  of  $K(2)$ . Without loss of generality, we can assume that  $M'$  is annihilated by  $\gamma$ . Since  $M$  is annihilated by  $\gamma$ , it is just a regular representation of the 2-Kronecker quiver with arrow basis  $\alpha$  and  $\beta$ . But any 4-dimensional regular representation of a 2-Kronecker quiver has a 2-dimensional regular submodule. This shows that  $M$  is not elementary. Altogether we have shown that the restriction of  $M$  to any 2-Kronecker subalgebra has 2-dimensional endomorphism ring.

If  $u$  is a non-zero element of  $M_1$ , then  $\Lambda u$  contains  $M_2$  and  $\dim \Lambda u = 3$ . Namely, if  $\Lambda u$  is of dimension 1, then  $\Lambda u$  simple injective, thus  $M$  cannot be indecomposable. If  $\Lambda u$  is of dimension 2, then  $\Lambda u$  is a proper non-zero regular submodule and then  $M$  is not elementary. It follows that  $\dim \Lambda u = 3$  and that  $M_2 \subset \Lambda u$ . Given any non-zero element  $u \in M_1$ , there is a non-zero element which annihilates  $u$ , say  $0 \neq \beta \in A$ . No element in  $A \setminus \langle \beta \rangle$  annihilates  $u$ , since otherwise the dimension of  $\Lambda u$  is at most 2. Let  $u, v$  be a basis of  $M_1$ . Let  $\beta, \gamma$  be non-zero elements of  $A$  with  $\beta(u) = 0, \gamma(v) = 0$ . Then the elements  $\beta, \gamma$  are linearly independent, since otherwise we would have  $\gamma(u) = 0$ , thus the submodule  $\Lambda u$  would be of dimension at most 2. The elements  $\beta(v), \gamma(u)$  must be linearly independent, since otherwise the restriction of  $M$  to  $\beta, \gamma$  would be the direct sum of a simple projective and an indecomposable injective. We take  $\beta(v), \gamma(u)$  as an ordered basis of  $M_2 = k^2$ , so that  $\beta(v) = (1, 0)$  and  $\gamma(u) = (0, 1)$ . Choose an element  $\alpha \in A \setminus \langle \beta, \gamma \rangle$ , thus  $\alpha, \beta, \gamma$  is a basis of  $A$ . Let  $\alpha(u) = (\kappa, \lambda)$  and  $\alpha(v) = (\mu, \nu)$  with  $\kappa, \lambda, \mu, \nu$  in  $k$ . Since  $\alpha(u)$  cannot be a multiple of  $\gamma(u) = (0, 1)$ , we see that  $\kappa \neq 0$ . Since  $\alpha(v)$  cannot be a multiple of

$\beta(v) = (1, 0)$ , we see that  $\nu \neq 0$ . Let  $\alpha' = \alpha - \mu\beta - \lambda\gamma$ . Then

$$\begin{aligned}\alpha'(u) &= \alpha(u) - \mu\beta(u) - \lambda\gamma(u) = (\kappa, \lambda) - (0, 0) - \lambda(0, 1) = (\kappa, 0), \\ \alpha'(v) &= \alpha(v) - \mu\beta(v) - \lambda\gamma(v) = (\mu, \nu) - \mu(1, 0) - (0, 0) = (0, \nu).\end{aligned}$$

Let  $\beta' = \kappa\beta$  and  $\gamma' = \nu\gamma$ . Then we have

$$\begin{aligned}\beta'(u) &= (0, 0), & \beta'(v) &= (\kappa, 0), \\ \gamma'(u) &= (0, \nu), & \gamma'(v) &= (0, 0),\end{aligned}$$

Altogether, we see that

$$\begin{array}{ccc} u & & v \\ \alpha' \downarrow & \nearrow \gamma' & \searrow \beta' \\ & & \downarrow \alpha' \\ (\kappa, 0) & & (0, \nu) \end{array}$$

Since both elements  $\kappa$  and  $\nu$  are non-zero, the elements  $(\kappa, 0)$  and  $(0, \nu)$  form a basis of  $k^2$ , and  $\alpha', \beta', \gamma'$  form a basis of  $A$ . Thus  $M$  is isomorphic to  $X(\alpha', \beta', \gamma')$ . This completes the proof.

**Lemma 4.2.** *A tree module with dimension vector  $(2, 2)$  cannot be elementary.*

Proof. If  $M$  is a tree module with dimension vector  $(2, 2)$ , then the coefficient quiver has to be of the form

$$\begin{array}{cc} \bullet & \bullet \\ \downarrow & \nearrow \\ \bullet & \bullet \end{array}$$

But then  $M$  has a submodule  $U$  such that both  $U$  and  $M/U$  have dimension vector  $(1, 1)$ .

**Lemma 4.3.** *If  $M$  is indecomposable with dimension vector  $(2, 2)$  and not elementary, then  $M$  is of one of the following forms*

$$\begin{array}{cc} \bullet & \bullet \\ \alpha \downarrow & \nearrow \beta \\ \bullet & \bullet \end{array} \quad \begin{array}{cc} \bullet & \bullet \\ \alpha \downarrow & \nearrow \beta \\ \bullet & \bullet \end{array}$$

for some basis  $\alpha, \beta, \gamma$  of  $A$ .

Proof. Let  $M$  be indecomposable with dimension vector  $(2, 2)$ . If  $M$  is not faithful, say annihilated by  $0 \neq \gamma \in A$ , then  $M$  is a  $K(2)$ -module and therefore as shown on the right.

Now assume that  $M$  is faithful, and not elementary. Since  $M$  is not elementary, there is an element  $0 \neq u \in M_1$  such that  $\Lambda u$  has dimension vector  $(1, 1)$ . The annihilator  $B$  of  $u$  is a 2-dimensional subspace of  $A$ . Let  $v \in M_1 \setminus \langle u \rangle$ . Since  $M$  is indecomposable, we see that  $\Lambda v$  has to be 3-dimensional and there is a non-zero element  $\alpha \in A$  with  $\alpha v = 0$ . Since  $M$  is faithful,  $\alpha(u) \neq 0$ . Also, since  $M$  is faithful, we have  $Bv = M_2$ . Thus, there is  $\beta \in B$  with  $\beta(v) = \alpha(u)$ . Let  $\gamma \in B \setminus \langle \beta \rangle$ . Then  $\alpha(u), \beta(\gamma)$  is a basis of  $M_2$ . With respect to the



basis  $\alpha, \beta, \gamma$  of  $A$ , the basis  $u, v$  of  $M_1$  and the basis  $\alpha(u), \beta(\gamma)$  of  $M_2$ , the module  $M$  has the form as depicted on the left.  $\square$ .

### 5. The structure of the modules $\sigma^t X(\alpha, \beta, \gamma)$ .

The 3-Kroncker modules  $I_i = \sigma^i S(2)$  are the preinjective modules, see [FR1].

**Proposition.** *For  $t \geq 1$ , there is an exact sequence*

$$0 \rightarrow X(\alpha, \beta, \gamma) \rightarrow \sigma^t X(\alpha, \beta, \gamma) \rightarrow \bigoplus_{0 \leq i < t} I_i^2 \rightarrow 0.$$

Proof. First we consider the case  $t = 1$ . There is an obvious embedding of  $X(\alpha, \beta, \gamma)$  into  $Y(\alpha, \beta, \gamma) = \sigma X(\alpha, \beta, \gamma)$ , thus there is an exact sequence of the form

$$0 \rightarrow X(\alpha, \beta, \gamma) \rightarrow Y(\alpha, \beta, \gamma) \rightarrow S(2)^2 \rightarrow 0.$$

Now we use induction. We start with the sequence

$$0 \rightarrow X(\alpha, \beta, \gamma) \rightarrow \sigma^t X(\alpha, \beta, \gamma) \rightarrow \bigoplus_{0 \leq i < t} I_i^2 \rightarrow 0,$$

for some  $t \geq 1$  and apply  $\sigma$ . In this way, we obtain the sequence

$$0 \rightarrow \sigma X(\alpha, \beta, \gamma) \rightarrow \sigma^{t+1} X(\alpha, \beta, \gamma) \rightarrow \bigoplus_{1 \leq i \leq t} I_i^2 \rightarrow 0.$$

This shows that  $M = \sigma^{t+1} X(\alpha, \beta, \gamma)$  has a submodule  $U$  isomorphic to  $\sigma X(\alpha, \beta, \gamma)$ , with  $M/U$  isomorphic to  $\bigoplus_{1 \leq i \leq t} I_i^2$ . But the case  $t = 1$  shows that  $U$  has a submodule  $U'$  isomorphic to  $X(\alpha, \beta, \gamma)$  with  $U/U'$  isomorphic to  $S(2)^2 = I_0^2$ . The embedding of  $U/U'$  into  $M/U'$  has to split, since  $I_0$  is injective. This completes the proof.

### Appendix 1. Elementary modules.

According to [K], Proposition 4.4, a regular representation  $M$  is elementary if and only if for any nonzero regular submodule  $U$  of  $M$ , the factor module  $M/U$  is preinjective. Let us include the proof of a slight improvement of this criterion.

We deal with the general setting where  $\Lambda$  is a hereditary artin algebra.

**Proposition.** *Let  $M$  be non-zero regular module  $M$ . Then  $M$  is elementary if and only if given any submodule  $U$  of  $M$ , the submodule  $U$  is preprojective or the factor module  $M/U$  is preinjective.*

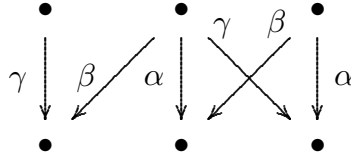
Proof. Let  $M$  be non-zero and regular. First, assume that for any submodule  $U$  of  $M$ , the submodule  $U$  is preprojective or the factor module  $M/U$  is preinjective. Then  $M$  cannot be a proper extension of regular modules, thus  $M$  is elementary.

Conversely, let  $M$  be elementary. Let  $U$  be a submodule which is not preprojective. Since  $M$  has no non-zero preinjective submodules, we can write  $U = U_1 \oplus U_2$  with  $U_1$  preprojective and  $U_2$  regular. Since  $U$  is not preprojective, we know that  $U_2$  is non-zero. Since  $M$  has no non-zero preprojective factor modules, we decompose  $M/U_2$  as a direct sum of a regular and a preinjective module: there are submodules  $V_1, V_2$  of  $M$  with  $V_1 \cap V_2 = U, V_1 + V_2 = M$  (thus  $M/U = V_1/U_2 \oplus V_2/U_2$ ) such that  $V_1/U_2$  is regular, and  $V_2/U_2$  is preinjective.

Consider  $V_2$ . First of all,  $V_2 \neq 0$ , since  $U_2$  is a non-zero submodule of  $V_2$ . Second, we claim that  $V_2$  is regular. Namely,  $V_2$  is an extension of the regular module  $U_2$  by the preinjective module  $V_2/U_2$ , thus it has no non-zero preprojective factor module. Thus, we can decompose  $V_2 = W_1 \oplus W_2$  with  $W_1$  regular,  $W_2$  preinjective. But  $W_2$  is a preinjective submodule of  $M$ , therefore  $W_2 = 0$ . This shows that  $V_2 = W_1$  is regular.

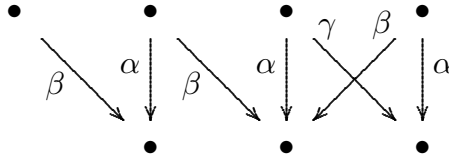
On the other hand,  $W/V_2$  is isomorphic to  $V_1/U_2$ , thus regular. But since  $M$  is not a proper extension of regular modules, it follows that  $W/V_2 = 0$ , thus  $V_2 = M$ . Therefore  $M/U_2 = V_2/U_2$  is preinjective. But  $M/U = M/(U_1 + U_2)$  is a factor module of  $M/U_2$ , and a factor module of a preinjective module is preinjective. This shows that  $M/U$  is preinjective.  $\square$

The definition of an elementary module implies that any regular module has a filtration by elementary modules. But such filtrations are not at all unique. This is well-known, but we would like to mention that the 3-Kronecker modules provide examples which are easy to remember. Here is the first such example  $M$ :



On the right we see that  $X(\alpha, \beta, \gamma)$  is a factor module, and the corresponding kernel is  $B(\gamma)$  (it is generated by the first base vector of  $M_1$ ). On the other hand, on the left we see that  $V(\beta, \gamma)$  is a factor module, and the corresponding kernel has dimension vector  $(1, 2)$  (it is generated by the last base vector of  $M_1$ ).

Here is the second example  $N$ :



On the right we see again that  $X(\alpha, \beta, \gamma)$  is a submodule, and the corresponding factor module is  $V(\alpha, \beta)$  (generated by the first two base vectors of  $N_1$ ). On the other hand, going from left to right, we see that the module has a filtration whose lowest two factors are of the form  $B(\beta)$ , whereas the upper factor is  $V(\alpha, \gamma)$  (generated by the last two base vectors of  $N_1$ ).

## Appendix 2: The representations of the 2-Kronecker quiver.

**Proposition.** *Any indecomposable  $K(2)$ -module is a tree module (with respect to some basis of the arrow space of  $K(2)$ ), and its coefficient quiver is of type  $\mathbb{A}$ .*

Proof. The preprojective and the preinjective modules are exceptional modules, thus they are tree modules with respect to any basis. The remaining indecomposable representations of  $K(2)$  are of the form  $R[t]$  where  $R$  is simple regular, and  $R[t]$  denotes the indecomposable regular module of dimension  $2t$  with regular socle  $R$ . We may choose a basis of the arrow space such that  $R$  is isomorphic to  $(k, k; 1, 0)$ . Then  $R[t]$  is a tree module such that the underlying graph of the coefficient quiver is of type  $\mathbb{A}_{2t}$ .

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